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PERIODS AND LIFTINGS: FROM G_2 TO C_3

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ABSTRACT

In this paper we characterize the exceptional theta lifting from G_2 to C_3 by means of the existence of the simple pole at $s = 1$ of the spin L-function for GSp_3 and by means of the nonvanishing of certain period on GSp_3. Among other results, we also prove the lifting from G_2 to C_3 is functorial at local unramified places.

1. Introduction

In the modern theory of automorphic forms, one of the basic problems is to study various relations of automorphic representations of different groups. Langlands' principle of functoriality asserts the relation (functorial lifting) of automorphic representations measured by relevant L-functions. However, there are many examples of liftings which are not functorial in the sense of Langlands. It seems also important to measure those non-functorial liftings. It is well known that

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the Arthur–Selberg trace formula method is a general approach to establish endoscopy (and/or the twisted version) liftings of automorphic representations. By contrast, the relative trace formula method which was initiated by Jacquet provides an alternative way to establish various different liftings of automorphic representations of particular types, which are usually characterized by means of periods. From this point of view, it is fundamental to study various models of a given automorphic representation and model-comparison relations of automorphic representations under a given lifting.

In this paper, we shall illustrate this idea by studying the theta correspondences between automorphic representations of $G_2(\mathbb{A})$ and those of $GSp_3(\mathbb{A})$ and studying the parabolic-induction from $GSp_3(\mathbb{A})$ to $F_4(\mathbb{A})$. We remark that the liftings from G_2 to GSp_3 and from GSp_3 to F_4 are known or proved in this paper to be functorial, but the lifting from GSp_3 to G_2 is not functorial.

The theta liftings between automorphic representations of $G_2(\mathbb{A})$ and $GSp_3(\mathbb{A})$ are constructed concretely by using the automorphic theta representation Θ_{GE_7}. Some fundamental work related to such liftings has been done globally in [GRS1] and [GRS2], and locally in [MS] and [L]. The lifting given by the parabolic induction from $GSp_3(\mathbb{A})$ to $F_4(\mathbb{A})$ is basically the correspondence from the cuspidal data (GSp_3, τ) to a certain residual representation of F_4. The basic results of this paper are the identities which relate certain models of a cuspidal representation of $G_2(\mathbb{A})$ to certain models of a cuspidal representation of $GSp_3(\mathbb{A})$ in the correspondence between G_2 and GSp_3 (Theorem 3.2 and 3.4), and relate the 'outer' period of the residual representation of F_4 to the 'inner' period of the cuspidal data (GSp_3, τ) (Theorem 4.1). As a consequence of Theorem 3.4, the local functoriality at unramified places of the lifting from G_2 to GSp_3 (Theorem 3.5) is proved in general, which was proved in [MS] under the assumption that the relevant representations are tempered. Moreover, the following is the main result of this paper (Theorem 3.6).

THEOREM 1.1 (Main): *Let π be an irreducible generic cuspidal automorphic representation of $GSp_3(\mathbb{A})$ with trivial central character. Then the following three statements are equivalent:*

(1) *$\pi \subset \Theta_{GE_7}^{GSp_3}(\sigma)$ for some irreducible generic cuspidal automorphic representation σ of $G_2(\mathbb{A})$.*

(2) *The partial spin L-function $L^S(\pi, Spin(7), s)$ (see (3.12) or [BG] for a definition) has a simple pole at $s = 1$.*

(3) *The period $\phi^{GL^{\Delta}(2);\psi_{N_2}}(g)$ is nonzero for some $\phi \in \pi$.*

The notations used here will be explained in §3. The analogy of the main result for the lifting from $PGL(3)$ to G_2 is stated as Theorem 3.7 without proof. It is expected that by the argument of [Jng1], the reductive periods in Theorem 4.1 should be related to the residual at $s = 1$ of the spin L-function $L(\pi, Spin(7), s)$. Moreover, by using the nonsplit forms of $Spin(8)$, these periods should be related to those studied by Gross and Savin in [GS] to construct the motives with G_2 as motivic Galois group.

The paper is organized as follows. We recall briefly in §2 the basics of the automorphic theta representation of GE_7 from [GRS1], [GRS2], and [Gur]. In §3, we first prove the cuspidality of theta lifting from GSp_3 to G_2 (Theorem 3.1). We only give a sketch of the proof, since it is basically the same proof as that for Theorem 3.2 in [GRS2]. Then we prove the model-comparison identities for the theta correspondence between G_2 and GSp_3 (Theorems 3.2 and 3.4) and prove the main result (Theorem 3.6). The main result in §4 is Theorem 4.1, which gives the model-comparison identity for the parabolic-induction from GSp_3 to F_4. The proof of Theorem 4.1 is given under assumption 4.1. More details about the assumption will be given near the end of §4. The proof of the convergence of integrals involved in §4.4 should follow from the same discussion as that given in [Jng].

Notations: We use the standard notations in this paper. The only thing we would like to point out is that for a split algebraic group of type X, X_r denotes the group with rank r, and for a classical group, we use notation $Sp_3 = Sp(6)$, for instance, to indicate the rank of the group and the size of the matrices realizing the group, respectively.

2. Automorphic theta representation of GE_7

We use standard notations for algebraic groups following [B]. The exceptional group E_7 has the following Dynkin diagram:

$$\begin{array}{ccccccccccccc} \alpha_1 & -- & \alpha_3 & -- & \alpha_4 & -- & \alpha_5 & -- & \alpha_6 & -- & \alpha_7 \\ & & & & | & & & & & & \\ & & & & \alpha_2 & & & & & & \end{array}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_7$ are the simple roots of E_7. For each α_i $(i = 1, 2, \ldots, 7)$, there is a well defined embedding from the group $SL(2)$ to the split group E_7, which takes torus elements $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ in $SL(2)$ to torus elements $h_i(t)$ in E_7. The maximal split torus of E_7 generated by $h_i(t_i)$ $(i = 1, 2, \ldots, 7)$ is denoted by

T, i.e.

$$T = \{h(t_1, \ldots, t_7) = \prod_{i=1}^{7} h_i(t_i) \mid t_i \in F^\times\}.$$

Let GE_7 denote the similitude group of the exceptional group of type E_7. One can define GE_7 by embedding E_7 into E_8, so that the similitude element of GE_7 is given by a one-dimensional torus in E_8 which acts linearly on the root subgroup attached to α_2 and trivially on the root subgroup attached to any other simple root. This distinguished one-dimensional torus is denoted by $h_8(t_8)$. Hence the maximal split torus of GE_7 is of dimension eight and its elements are expressed as:

$$h(t_1, \ldots, t_8) = \prod_{i=1}^{8} h_i(t_i), \quad t_i \in F^\times.$$

If $\alpha = \sum_{i=1}^{7} n_i\alpha_i$, we shall denote the root α simply by $\alpha = (n_1n_2\cdots n_7)$. We denote by x_α the one-parameter additive subgroup attached to the root α. We denote by w_i the Weyl group element representing the simple reflection with respect to the simple root α_i and we denote $w[ijk\cdots l] = w_iw_jw_k\cdots w_l$. The basic properties for GE_7 may be found in [G1] and [Gur].

In the papers [GRS1] and [GRS2] the definition and basic properties of the automorphic theta representation on simply laced, split groups were studied. The extension of these results to GE_7 was done in [Gur]. In the following we recall some basic facts on the automorphic theta representation of GE_7 from [Gur].

Let $P(E_6) = M(E_6)V^{(7)}$ denote the parabolic subgroup of GE_7 whose Levi part $M(E_6)$ contains the group E_6 and $V^{(7)}$ is abelian consisting of all the positive roots $\alpha = \sum_{i=1}^{7} n_i\alpha_i$ with $n_7 = 1$. As usual, to any section f_s in the (normalized) induced representation $Ind_{P(E_6)(\mathbb{A})}^{GE_7(\mathbb{A})}(\delta_{P(E_6)}^s)$ we attach an Eisenstein series

$$E(g,s) = \sum_{\gamma \in P(E_6)(F)\backslash GE_7(F)} f_s(\gamma g),$$

which converges absolutely for $Re(s)$ large and has a meromorphic continuation to $\mathbb{C}$.

Proposition 2.1 ([GRS1],[GRS2] and [Gur]): *The Eisenstein series $E(g,s)$ has the following properties:*

(1) *$E(g,s)$ has at most a simple pole at $s = 5/18$ and the residue can be achieved by the spherical section.*
(2) *The residue $res_{s=5/18}E(g,s)$ is square-integrable.*

(3) *The irreducible component Θ_{GE_7} of $\mathrm{res}_{s=5/18}E(g,s)$ generated by the spherical section is called the automorphic theta representation of GE_7. Any automorphic function in Θ_{GE_7}, which is denoted by θ_{ge_7}, has the following Fourier expansion:*

$$\theta_{ge_7}(g) = \theta_{ge_7}^{V^{(7)}}(g) + \sum_{\gamma \in Stab_{M(E_6)}(\psi_{V^{(7)}})\backslash M(E_6)(F)} \theta_{ge_7}^{\psi_{V^{(7)}}}(\gamma g), \tag{2.1}$$

where the constant term $\theta_{ge_7}^{V^{(7)}}(g)$ is given by

$$\theta_{ge_7}^{V^{(7)}}(g) = \int_{V^{(7)}(F)\backslash V^{(7)}(\mathbb{A})} \theta_{ge_7}(vg)dv$$

and the non-trivial Fourier coefficient $\theta_{ge_7}^{\psi_{V^{(7)}}}(g)$ is given by

$$\theta_{ge_7}^{\psi_{V^{(7)}}}(g) = \int_{V^{(7)}(F)\backslash V^{(7)}(\mathbb{A})} \theta_{ge_7}(vg)\psi_{V^{(7)}}(v)dv.$$

The unitary additive character $\psi_{V^{(7)}}$ of $V^{(7)}(F)\backslash V^{(7)}$ is defined by

$$\psi_{V^{(7)}}(v) = \psi_{V^{(7)}}(x_{\alpha_7}(r)v') = \psi_0(r)$$

if one writes a general element $v \in V^{(7)}$ in the form $v = x_{\alpha_7}(r)v'$ (this is well defined since $V^{(7)}$ is abelian), where ψ_0 is a non-trivial unitary additive character of $\mathbb{A}/F$.

(4) *For all $r \in Stab_{M(E_6)}(\psi_{V^{(7)}})(\mathbb{A})$, the non-trivial Fourier coefficient $\theta_{ge_7}^{\psi_{V^{(7)}}}(g)$ has the property that*

$$\theta_{ge_7}^{\psi_{V^{(7)}}}(rg) = \theta_{ge_7}^{\psi_{V^{(7)}}}(g). \tag{2.2}$$

It is remarkable that the Fourier expansion of $\theta_{GE_7}(g)$ in (2.1), which reflects the minimality of the representation, is crucial in various applications of automorphic theta representations. The following is another useful Fourier expansion for the automorphic theta function θ_{ge_7} (§4.7 in [Gur]), which is over the maximal unipotent radical $V^{(1)}$, where $P(D_6) = M(D_6)V^{(1)}$ is the maximal parabolic subgroup with the Levi subgroup $M(D_6)$ containing $SO(12)$. Note that Θ_{GE_7} can also be realized as the unramified irreducible constituent of the residue at $s = 11/34$ of the Eisenstein series associated to sections in the normalized induced representation $Ind_{P(D_6)(\mathbb{A})}^{GE_7(\mathbb{A})}(\delta_{P(D_6)}^s)$. Since $V^{(1)}$ is of Heisenberg type with center $R = \{x_{2234321}(r)\}$, we have

(2.3)
$$\int_{\mathbb{A}/F} \theta_{ge_7}(x_{2234321}(r)g)dr = \theta_{ge_7}^{V^{(1)}}(g) + \sum_{\gamma \in Stab_{M(D_6)}(\psi_{V^{(1)}}) \backslash M(D_6)(F)} \theta_{ge_7}^{\psi_{V^{(1)}}}(\gamma g),$$

where $\theta_{ge_7}^{V^{(1)}}(g)$ is the constant term along $V^{(1)}$ and the non-trivial Fourier coefficient $\theta_{ge_7}^{\psi_{V^{(1)}}}(g)$ is defined over $V^{(1)}(F)\backslash V^{(1)}(\mathbb{A})$ with the unitary additive character $\psi_{V^{(1)}}$ attached to α_1 (see §4.7 of [Gur] for more details). Then for all $r \in Stab_{M(D_6)(\mathbb{A})}(\psi_{V^{(1)}})$

$$\theta_{ge_7}^{\psi_{V^{(1)}}}(rg) = \theta_{ge_7}^{\psi_{V^{(1)}}}(g). \tag{2.4}$$

Following from Theorem 4.6 in [Gur], as representation of $M^0(D_6)$, the representation $\Theta_{GE_7}^{V^{(1)}}$ generated by the constant term $\theta_{ge_7}^{V^{(1)}}(g)$ can be expressed as follows:

$$\Theta_{GE_7}^{V^{(1)}}|_{M^0(D_6)} = Triv_{SO(12)} \oplus \Theta_{SO(12)}, \tag{2.5}$$

where $M(D_6) = GL_1^2 \cdot M^0(D_6)$, $M^0(D_6) \simeq SO(12)$, $Triv_{SO(12)}$ is a representation of $M(D_6)$ where $SO(12)$ acts trivially, and $\Theta_{SO(12)}$ is the automorphic theta representation of $SO(12)$ defined in [GRS1].

3. Correspondences between G_2 and GSp_3

3.1 Dual Pair $G_2 \times GSp_3$ in GE_7. It is well known that $G_2 \times GSp_3$ forms a reductive dual pair in GE_7. One of the embeddings of $G_2 \times Sp_3$ in E_7 was given explicitly in [GRS2], which gives rise to an embedding of $G_2 \times GSp_3$ into GE_7 by embedding the similitude factor of GSp_3 into GE_7:

$$diag(a, a, a, 1, 1, 1) \mapsto h(a^{-1}, a^3, a^5, a^6, a^9, a^7, a^5, a^3),$$

where $h(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) = \prod_{i=1}^8 h_i(t_i)$ is a generic element in the maximal torus of GE_7. The embedding of the simple roots of GSp_3 are given by (see p. 459 of [GRS2])

$$\begin{aligned} x_{100}(r) &\mapsto x_{1011100}(r)x_{1111000}(r), \\ x_{010}(r) &\mapsto x_{0011110}(r)x_{0101110}(r), \\ x_{001}(r) &\mapsto x_{0101110}(r), \end{aligned} \tag{3.1}$$

and the simple positive roots a (short) and b (long) of G_2 are embedded as (see p. 545 of [GRS2])

$$(3.2)\qquad \begin{aligned} x_a(r) &\mapsto x_{0100000}(r)x_{0010000}(-r)x_{0000100}(r),\\ x_b(r) &\mapsto x_{0001000}(r). \end{aligned}$$

We call this embedding the G_2-embedding of $G_2 \times GSp_3$ into GE_7. By conjugating this embedding by the Weyl element $w[342134567432454361]$ (see p. 111 of [G1]), we obtain another embedding of $G_2 \times GSp_3$ into GE_7. The explicit embedding given below will be used later:

$$(3.3)\qquad \begin{aligned} x_{100}(r) &\mapsto x_{0100000}(r)x_{0000001}(-r),\\ x_{010}(r) &\mapsto x_{0001000}(r)x_{0000010}(-r),\\ x_{001}(r) &\mapsto x_{0000100}(r), \end{aligned}$$

and the simple roots of G_2 are embedded as (see p. 545 of [GRS2])

$$(3.4)\qquad \begin{aligned} x_a(r) &\mapsto x_{0111110}(r)x_{0011111}(m_1r)x_{0112100}(m_2r),\\ x_b(r) &\mapsto x_{1000000}(r), \end{aligned}$$

where $m_i = \pm 1$. We call this later embedding the C_3-embedding of $G_2 \times GSp_3$ into GE_7.

3.2 LIFTING FROM GSp_3 TO G_2. In this subsection, we assume that π is an irreducible cuspidal automorphic representation of $GSp_3(\mathbb{A})$ with trivial central character. The integral of the following type gives rise to the lifting,

$$(3.5)\qquad \theta_{ge_7}(h;\phi) := \int_{Z(\mathbb{A})\cdot GSp_3(F)\backslash GSp_3(\mathbb{A})} \phi(g)\theta_{ge_7}(h,g)dg,$$

where $\phi(g)$ is a cusp form in π and $\theta_{ge_7}(g)$ is an automorphic theta function in the automorphic theta representation Θ_{GE_7} of $GE_7(\mathbb{A})$, and Z is the center of GSp_3. We denote by $\Theta^{G_2}_{GE_7}(\pi)$ the space of automorphic functions on $G_2(\mathbb{A})$ generated by all the integrals in (3.5). The representation π is generic if

$$\int_{U_3(F)\backslash U_3(\mathbb{A})} \phi(ug)\psi_{U_3}(u)du \neq 0$$

for some $\phi \in \pi$, where U_3 is the maximal unipotent subgroup of GSp_3 and ψ_{U_3} is the generic character defined by

$$\psi_{U_3}(u) := \psi_0(r_1 + r_2 + r_3)$$

where $u = x_{100}(r_1)x_{010}(r_2)x_{001}(r_3)u' \in U_3$ and $u' \in [U_3, U_3]$, and 100, 010 and 001 indicate the three simple positive roots of GSp_3 as in (3.1).

THEOREM 3.1: *If π is generic, then the space $\Theta_{GE_7}^{G_2}(\pi)$ generated by all the automorphic functions in (3.5) is a cuspidal representation of $G_2(\mathbb{A})$.*

Proof: We shall sketch the proof here since it is basically the same as that of Theorem 3.2 in [GRS2]. We use the C_3-embedding of $G_2 \times GSp_3$ in GE_7 as described in (3.3) and (3.4) in §3.1. It will be enough to show that

$$\int_{U(F)\backslash U(\mathbb{A})} \theta_{ge_7}(u;\phi)du = 0$$

for all choices of data and the unipotent radical U of all maximal parabolic subgroups of G_2. As usual, it suffices to reduce the problem to the non-similitude case, i.e. to show

$$I = \int_{U(F)\backslash U(\mathbb{A})} \int_{Sp_3(F)\backslash Sp_3(\mathbb{A})} \phi(g)\theta_{e_7}(u,g)dgdu = 0$$

for all choices of data. Note that the restriction of the automorphic theta representation Θ_{GE_7} of GE_7 to E_7 is the automorphic theta representation Θ_{E_7} of E_7, i.e. $\theta_{ge_7}(x) = \theta_{e_7}(x)$ for $x \in E_7$.

Since the center $R = \{x_{2234321}(r)\}$ of the Heisenberg unipotent subgroup $V^{(1)}$ of E_7 is contained in the unipotent radical of any standard parabolic subgroup of G_2, we first factor the constant term along the center:

$$I = \int_{R(\mathbb{A})U(F)\backslash U(\mathbb{A})} \int_{Sp_3(F)\backslash Sp_3(\mathbb{A})} \phi(g) \int_{F\backslash\mathbb{A}} \theta_{e_7}(x_{2234321}(r)(u,g))drdgdu.$$

By applying the Fourier expansion (2.3), integral I becomes a sum of two integrals:

$$I_1 = \int_{R(\mathbb{A})U(F)\backslash U(\mathbb{A})} \int_{Sp_3(F)\backslash Sp_3(\mathbb{A})} \phi(g)\theta_{e_7}^{V^{(1)}}(u,g)dgdu,$$

$$I_2 = \int_{R(\mathbb{A})U(F)\backslash U(\mathbb{A})} \int_{Sp_3(F)\backslash Sp_3(\mathbb{A})} \phi(g) \sum_{\substack{\gamma\in Stab_{M(D_6)}(\psi_{V^{(1)}})\backslash M(D_6)(F)\\ \epsilon\in F^\times}} \theta_{ge_7}^{\psi_{V^{(1)}}}(h(\epsilon)\gamma(v,g))dgdv,$$

where $h(\epsilon) = h(1,1,\epsilon,1,1,1,1,1)$. We claim that both integrals I_1 and I_2 are zero for all choices of data.

For integral I_1, we use formula (2.5). Since Sp_3 is a subgroup of $SO(12)$ under the C_3-embedding, it reduces to show

$$\int_{Sp_3(F)\backslash Sp_3(\mathbb{A})} \phi(g)\theta_{SO(12)}(g)dg = 0$$

for all choices of data. Using the fact that the automorphic theta representation $\Theta_{SO(12)}$ is the image of the usual theta lifting of the trivial representation of $SL(2)$ (Formula (2.3) in [GRS2]), it is not hard to show that $I_1 = 0$.

To show that I_2 is zero we need to proceed as in the proof of Theorem 3.1 in [GRS2]. We start with the decomposition

$$\begin{aligned} P(A_5)\backslash SO(12) &= \bigcup_{\gamma\in P(A_5)\backslash SO(12)} P(A_5)\cdot\gamma \\ &= \bigcup_{\delta\in P(A_5)\backslash SO(12)/P(A_5)} \bigcup_{\mu\in\Gamma_\delta\backslash P(A_5)} P(A_5)\cdot\delta\cdot\mu \end{aligned}$$

where $\Gamma_\delta = \delta^{-1}P(A_5)\delta \cap P(A_5)$. Then by using (2.4), more double-coset decomposition and the character $\psi_{V^{(1)}}$, we can show that I_2 is zero for all choices of data. As we mentioned before we omit the details. ∎

If $\Theta_{GE_7}^{G_2}(\pi) \neq 0$, then the generic cuspidal representation π of $GSp_3(\mathbb{A})$ is lifted to some cuspidal representations of $G_2(\mathbb{A})$. To determine the nonvanishing condition, we need the following model-comparison theorem.

Let U_2 be the maximal unipotent subgroup of G_2. Then any element of U_2 has the form $u = x_a(r_a)x_b(r_b)u'$, where $u' \in [U_2, U_2]$. We define

$$\psi_{U_2}(u) = \psi_0(r_a + r_b),$$

which is a (generic) character of $U_2(F)\backslash U_2(\mathbb{A})$. On the other hand, we take the maximal parabolic subgroup $P_2 = M_2N_2$ of GSp_3 with the Levi factor $GL(2) \times GL(2)$ and the unipotent radical

$$N_2 = \{n(x,*) = \begin{pmatrix} I & x & * \\ & I & x^* \\ & & I \end{pmatrix}\} \subset GSp_3.$$

Define a character ψ_{N_2} on N_2 by

$$\psi_{N_2}(n(x,*)) := \psi_0(tr(x)),$$

where $tr(x)$ is the trace of x. The stabilizer of ψ_{N_2} in $GL(2)\times GL(2)$ is $GL^{\Delta}(2)$, the image of the diagonal embedding. Given $\phi \in \pi$ we define

$$\text{(3.6)} \quad \phi^{GL(2)^{\Delta};\psi_{N_2}}(g) := \int_{Z(\mathbb{A})[GL(2)^{\Delta}N_2](F)\backslash[GL(2)^{\Delta}N_2](\mathbb{A})} \phi(nmg)\psi_{N_2}(n)dndm.$$

This period integral played an important role in the study of the spin L-function $L(s,\pi,spin(7))$ ([BG], [V]).

THEOREM 3.2: *For all choices of data, the following identity holds:*

$$\int_{U_2(F)\backslash U_2(\mathbb{A})} \theta_{ge_7}(u;\phi)\psi_{U_2}(u)du$$
$$= \int_{[GL(2)^{\Delta}N_2](\mathbb{A})\backslash GSp_3(\mathbb{A})} \phi^{GL(2)^{\Delta};\psi_{N_2}}(g) \int_{\mathbb{A}} \theta_{ge_7}^{\psi_{V^{(1)}}}(\mu x_{0112100}(r)(1,g))\psi_0(r)drdg,$$

where $\mu := w[345243]x_{0111000}(1)x_{0011100}(1)$.

Proof: The idea to prove this model-comparison identity comes from the explicit computation in the proof of Theorem 3.1, which was omitted there. Following the same computation as sketched for Theorem 3.1, one ends up with

$$\int_{U_2(F)\backslash U_2(\mathbb{A})} \theta_{ge_7}(u;\phi)\psi_{U_2}(u)du$$
$$= \int_{Z(\mathbb{A})[GL(2)^{\Delta}N_2](F)\backslash GSp_3(\mathbb{A})} \phi(g)$$
$$\cdot \int_{(F\backslash\mathbb{A})^2} \sum_{\delta\in F;\epsilon\in F^{\times}} \theta_{ge_7}^{\psi_{V^{(1)}}}(h(\epsilon)\mu x_{0112100}(\delta)(n(r_1,r_2),g)\psi_0(r_1+r_2)dr_1dr_2dg,$$

where $n(r_1,r_2) := x_{1000000}(r_1)x_{0111110}(r_2)x_{0011111}(r_2)x_{0112100}(r_2)$.

From here one proceeds as follows. Since

$$w[345243]\cdot(1122100) = (1000000),$$
$$x_{0111000}(1)x_{0011100}(1)x_{1000000}(r_1) = x_{1122100}(r_1)yx_{0111000}(1)x_{0011100}(1),$$

where $y \in GE_7(\mathbb{A})$ with the property that

$$w[345243]yw[345243] \in Stab_{P(D_6)(\mathbb{A})}(\psi_{V^{(1)}}).$$

Using (2.4) we may ignore this element. Hence by conjugating $x_{1000000}(r_1)$ to the left we obtain $\int_{F\backslash\mathbb{A}}\psi((\epsilon-1)r_1)dr_1$ as inner integration. We must have $\epsilon = 1$, otherwise the integral vanishes. Hence the integration over the variable r_1 disappears.

Next, by conjugating $x_{0111110}(r_2)x_{0011111}(r_2)$ to the left and using (2.4), the summation over $\delta \in F$ with integration over $r_2 \in F\backslash\mathbb{A}$ the above integral equals

$$\int_{Z(\mathbb{A})[GL(2)^{\Delta}N_2](F)\backslash GSp_3(\mathbb{A})} \phi(g) \int_{\mathbb{A}} \theta_{ge_7}^{\psi_{V^{(1)}}}(\mu x_{0112100}(r)(1,g)\psi_0(r)drdg.$$

Factoring integration over g through $[GL(2)^{\Delta}N_2](F)\backslash[GL(2)^{\Delta}N_2](\mathbb{A})$ and using (2.4) again, we obtain the model-comparison identity. ∎

THEOREM 3.3: *If the representation π has the property that $\phi^{GL^{\Delta}(2);\psi_{N_2}}(g)$ does not vanish for some $\phi \in \pi$, then π has a nonzero lifting, i.e. $\Theta^{G_2}_{GE_7}(\pi) \neq 0$.*

Proof: By Theorem 3.2, we will prove a stronger result, that if $\phi^{GL^{\Delta}(2);\psi_{N_2}}(g)$ does not vanish for some $\phi \in \pi$, then

$$\int_{U_2(F)\backslash U_2(\mathbb{A})} \theta_{ge_7}(u;\phi)\psi_{U_2}(u)du \neq 0 \tag{3.7}$$

for a certain choice of data. Assume that (3.7) vanishes for all choices of data.

Let J_6 be the 6×6-matrix with 0-entries except the antidiagonal ones, where they are 1, and $M = \{m \in Mat_{6\times 6} : Jm^t + mJ = 0\}$. The group GSp_3 acts on M by $g\cdot m = gmJg^tJ$ (the one induced from the adjoint action of the Levi subgroup of $P(A_5)$ of $SO(12)$ on its unipotent radical). We define a character ψ_M on M by $\psi_M(m) = \psi_0(m_{51})$. By the C_3-embedding, M can be embedded in GE_7 as $\{x_\alpha(r)\}$ where α varies over the roots

$$\begin{array}{lllll} (1122100), & (1122110), & (1122111), & (1122210), & (1122211), \\ (1122221), & (1123210), & (1123211), & (1223210), & (1123221), \\ (1223211), & (1223221), & (1123321), & (1223321), & (1224321). \end{array}$$

Let $\Phi(m)$ be a Schwartz function on $M(\mathbb{A})$. From our vanishing assumption it follows from Theorem 3.2 that

$$\int_{M(\mathbb{A})}\int_{[GL(2)^{\Delta}N_2](\mathbb{A})\backslash GSp_3(\mathbb{A})} \phi^{GL(2)^{\Delta};\psi_{N_2}}(g) \cdot \int_{\mathbb{A}} \theta^{\psi_{V(1)}}_{GE_7}(\mu x_{0112100}(r)(1,g)m)\psi_0(r)\Phi(m)drdgdm$$

vanishes for all choices of data. By conjugating m to the left and using (2.4) this integral is equal to

$$\int_{[GL(2)^{\Delta}N_2\backslash GSp_3](\mathbb{A})} \phi^{GL(2)^{\Delta};\psi_{N_2}}(g) \cdot \int_{\mathbb{A}} \theta^{\psi_{V(1)}}_{GE_7}(\mu x_{0112100}(r)(1,g))\psi_0(r)\int_{M(\mathbb{A})}\Phi(g^{-1}\cdot m)\psi_M(m)dmdrdg.$$

Changing variables in m we obtain

$$\int_{[GL(2)^{\Delta}N_2](\mathbb{A})\backslash GSp_3(\mathbb{A})} \phi^{GL(2)^{\Delta};\psi_{N_2}}(g) \cdot \int_{\mathbb{A}} \theta^{\psi_{V(1)}}_{GE_7}(\mu x_{0112100}(r)(1,g))\psi_0(r)\hat{\Phi}(g)drdg,$$

where

$$\hat{\Phi}(h) = \int_{M(\mathbb{A})} \Phi(m)\psi_M(h \cdot m)dm.$$

Since the stabilizer of ψ_M under the g-action in GSp_3 is $[GL(2) \times GL(2)]^\circ N_2$, where

$$[GL(2) \times GL(2)]^\circ = \{(g_1, g_2) \in GL(2) \times GL(2)\colon \det g_1 = \det g_2\},$$

and since $\hat{\Phi}(g)$ is an arbitrary Schwartz function in $[GL(2) \times GL(2)]^\circ N_2 \backslash GSp_3$, the vanishing assumption implies that the integral

$$\int_{[GL(2)^\Delta \backslash [GL(2)\times GL(2)]^\circ](\mathbb{A})} \int_{\mathbb{A}} \phi^{GL(2)^\Delta;\psi_{N_2}}(g)\theta_{GE_7}^{\psi_{V(1)}}(\mu x_{0112100}(r)(1,g))\psi_0(r)drdg$$

vanishes for all choices of data.

Now using the root (1010000) we may get rid of the integral over r. Using the roots (1011000); (1111000); (1011100); (1111100) on which $[GL(2) \times GL(2)]^\circ$ acts as the tensor product, we derive that

$$\phi^{GL(2)^\Delta;\psi_{N_2}}(e)\theta_{GE_7}^{\psi_{V(1)}}(\mu)$$

is zero for all choices of data. This contradicts our assumption. ∎

3.3. Lifting From G_2 to GSp_3. The lifting from G_2 to GSp_3 has been studied in [GRS2]. We establish here a model-comparison formula, which leads to a proof of local unramified functoriality of this lifting in general. For tempered representations, the local unramified functoriality of this lifting was verified in [MS].

Let σ be an irreducible cuspidal automorphic representation of $G_2(\mathbb{A})$. We say σ is generic if

$$W_\varphi(h) = \int_{U_2(F)\backslash U_2(\mathbb{A})} \varphi(uh)\psi_{U_2}(u)du \neq 0$$

for some $\varphi \in \sigma$, where U_2 is the maximal unipotent subgroup of G_2 and ψ_{U_2} is the generic character defined in §3.2. Consider

$$\theta_{ge_7}(g;\varphi) := \int_{G_2(F)\backslash G_2(\mathbb{A})} \varphi(h)\theta_{ge_7}(h,g)dh, \tag{3.8}$$

where $\varphi(h)$ is a cusp form in σ and $\theta_{ge_7}(x)$ is an automorphic theta function in the automorphic theta representation Θ_{GE_7} of $GE_7(\mathbb{A})$. We denote by $\Theta_{GE_7}^{GSp_3}(\sigma)$ the space of automorphic functions on $GSp_3(\mathbb{A})$ generated by all integrals in (3.8).

The model-comparison identity in this case is

THEOREM 3.4: *If σ is an irreducible generic cuspidal automorphic representation of $G_2(\mathbb{A})$, then the identity*

$$(3.9)\quad \int_{U_3(F)\backslash U_3(\mathbb{A})} \theta_{ge_7}(ug,\varphi)\psi_{U_3}(u)du = \int_{U_2(\mathbb{A})\backslash G_2(\mathbb{A})} W_\varphi(h) \cdot \int_{\mathbb{A}^3} \theta_{ge_7}^{\psi_{V^{(7)}}}(\nu x_{1112110}(r)(h, x_{010}(x_1)x_{110}(x_2)g)\psi_0(r+x_1)drdx_1dx_2dh$$

holds for all choices of data, where $\nu = w[654234561]x_{0001110}(1)x_{1111110}(1)$, $\varphi \in \sigma$, and $\theta_{ge_7} \in \Theta_{GE_7}$.

Proof: The computation to obtain the model-comparison identity is the same as that in Theorem 3.2. We just mention that we use here the G_2-embedding of $G_2 \times GSp_3$ in GE_7 as described in (3.1) and (3.2) in §3.1. ∎

Next we shall apply Theorem 3.4 to verify the local unramified Langlands functoriality of this lifting. Note that as Theorem 6.2 in [GRS1] and Proposition 4.8.1 in [Gur], one has the local uniqueness of the $\psi_{V^{(7)}}$-quasi-invariant functional on the local component of Θ_{GE_7}. This local uniqueness implies the eulerian factorizability of the right hand side of (3.9). It is this observation that leads to the proof of the local unramified Langlands functoriality of this lifting.

Let F be a local field from now on till the end of this subsection. From [GRS1] it follows that the local component of the automorphic theta representation Θ_{GE_7} is a subrepresentation of $Ind_{P(E_6)}^{GE_7}(\delta_{P(E_6)}^{-5/18})$ (in the image of the standard intertwinning operator). Given a function f in $Ind_{P(E_6)}^{GE_7}(\delta_{P(E_6)}^{-5/18})$, we consider the functional defined by

$$(3.10)\qquad \mathcal{L}(x \cdot f) := \int_F f(w[7]x_{0000001}(r)x)\psi_0(r)dr,$$

where $x \in GE_7$. Clearly it satisfies the same properties as the local analogy of the $\psi_{V^{(7)}}$-Fourier coefficient $\theta_{ge_7}^{\psi_{V^{(7)}}}(x)$. From this it follows that the local integral corresponding to the right hand side of (3.9) is given by

$$(3.11)\quad \int_{U_2\backslash G_2} W(h) \cdot \int_{F^3} \mathcal{L}(\nu x_{1112110}(r_2)(h, x_{010}(x_1)x_{110}(x_2)g)\cdot f)\psi_0(r_2+x_1)dr_2dx_1dx_2dh,$$

where $W(h)$ is a local Whittaker function associated to the cuspidal automorphic representation σ of $G_2(\mathbb{A})$. Assume that the local component (we also use the notation σ) of σ is unramified. Then it is a constituent of $Ind_{B_2}^{G_2}(\chi)$, where B_2 is the Borel subgroup of G_2 and

$$\chi(diag(ab, a, b, 1, b^{-1}, a^{-1}, a^{-1}b^{-1})) = \chi_1(a)\chi_2(b),$$

where χ_i are unramified characters of $F^\times$ and $diag(ab, a, b, 1, b^{-1}, a^{-1}, a^{-1}b^{-1})$ is the image of the maximal split torus of G_2 embedded into $SO(7)$. Every unramified representation of GSp_3 is a constituent of $Ind_{B_3}^{GSp_3}(\eta)$, where B_3 is the Borel subgroup of GSp_3 and

$$\eta(diag(ab, ac, ad, d^{-1}, c^{-1}, b^{-1})) = \nu(a)\eta_1(b)\eta_2(c)\eta_3(d),$$

where $\nu(a), \eta_1(b), \eta_2(c), \eta_3(d)$ are unramified characters of $F^\times$. Given χ as above we define a character $\eta(\chi)$ of B_3 by setting $\nu = 1$, $\eta_1 = \chi_1$, $\eta_2 = \chi_2$ and $\eta_3 = \chi_1\chi_2$. Let σ be as above and let π be an unramified irreducible representation contained in the image of the local theta correspondence $\Theta_{GE_7}^{GSp_3}(\sigma)$. Then we have

THEOREM 3.5: *Assume that F is a p-adic field. With the notations as above, π is the irreducible unramified constituent of $Ind_{B_3}^{GSp_3}(\eta(\chi))$.*

Proof: We denote by p the local uniformizer of F. Let

$$t_\sigma = diag(\chi_1\chi_2(p), \chi_1(p), \chi_2(p), 1, \chi_2^{-1}(p), \chi_1^{-1}(p), \chi_1^{-1}\chi_2^{-1}(p))$$

be a representative of the semisimple conjugacy class in $G_2(\mathbb{C})$ associated to σ, embedded into $SO(7, \mathbb{C})$. We define the local standard L-factor

$$L(\sigma, St, s) = \det(I_7 - q^{-s}t_\sigma)^{-1}.$$

Let π be the irreducible unramified constituent in $Ind_{B_3}^{GSp_3}(\eta)$ and let

$$t_\pi = diag(\nu\eta_1\eta_2\eta_3(p), \nu\eta_1(p), \nu\eta_2(p), \nu\eta_3(p), \eta_3^{-1}(p), \eta_2^{-1}(p), \eta_1^{-1}(p), \eta_1^{-1}\eta_2^{-1}\eta_3^{-1}(p))$$

be a representative of the semisimple conjugacy class in $GSpin(7, \mathbb{C})$ (the L-group of GSp_3) associated to π embedded into $GO(8, \mathbb{C})$. We define as in [BG] the local spin L-factor

$$L(\pi, Spin(7), s) = \det(I_8 - q^{-s}t_\pi)^{-1}. \tag{3.12}$$

To prove our result, it suffices to show that

$$L(\pi, Spin(7), s) = \zeta(s)L(\sigma, St, s)$$

where $\zeta(s) = (1 - q^{-s})^{-1}$. In order to do so, we use the following local integral computation.

It follows from [G2] that

$$\int_{F^\times} W_\sigma(h(a,1))|a|^{s-3}d^\times a = \frac{L(\sigma, St, s)}{\zeta(2s)},$$

where $h(a,b) = diag(ab, a, b, 1, b^{-1}, a^{-1}, a^{-1}b^{-1})$ is the maximal torus of G_2. On the other hand, for π we let W_π be the unramified Whittaker function (normalized so that $W_\pi(e) = 1$) on GSp_3. Then it follows from [BG] that

$$\int_{F^\times} W_\pi(diag(\gamma,\gamma,\gamma,1,1,1))|\gamma|^{s-3}d^\times\gamma = \frac{L(\pi, Spin(7), s)}{\zeta(2s)}.$$

Thus it is enough to prove that

$$\int_{F^\times} W_\pi(diag(\gamma,\gamma,\gamma,1,1,1))|\gamma|^{s-3}d^\times\gamma = \zeta(s)\int_{F^\times} W_\sigma(h(a,1))|a|^{s-3}d^\times a \tag{3.13}$$

if π is the local lifting of σ under $\Theta_{GE_7}^{GSp_3}$.

Now take the integral in (3.11) to be $W_\pi(g)$ normalized so that $W_\pi(e) = 1$ (assume that f is unramified in the local component of Θ_{GE_7}). According to the G_2-embedding as in (3.1) and (3.2), we have

$$\begin{aligned} diag(\gamma,\gamma,\gamma,1,1,1) &\mapsto h(\gamma^{-1},\gamma^3,\gamma^5,\gamma^6,\gamma^9,\gamma^7,\gamma^5,\gamma^3) := t(\gamma), \\ h(a,b) &\mapsto h(1,1,ab,ab,a^2b,ab,1,1,1) := t(a,b). \end{aligned}$$

Applying the Iwasawa decomposition to the integral in (3.11), we obtain

$$\begin{aligned} \int_{[F^\times]^2} W_\sigma(h(ab,b)) \int_{F^3} \mathcal{L}(\nu x_{1112110}(r_2)(1, x_{010}(x_1)x_{110}(x_2))t(\gamma)t(a,b)\cdot f) \\ \cdot \psi_0(r_2 + x_1)|a^6b^{10}|^{-1}dr_2dx_1dx_2d^\times(a,b). \end{aligned}$$

Let $w' = w[7]w_0$ where $w_0 = w[654234561]$. Since

$$w't(\gamma)t(a,b)[w']^{-1} = h(\gamma^{-1},\ldots,\gamma^2ab^2),$$

and

$$\delta_{P(E_6)}^{-\frac{5}{18}+\frac{1}{2}}(h(t_8,t_1,\cdot,t_7)) = |t_7^4t_8^6|,$$

we get a factor $|\gamma^2a^4b^8|$. We also get a factor $|\gamma a^{-1}b^{-3}|$ from the change of variables in the additive variables. Finally, we conjugate $x_{0000001}(r_1)$ to the right and obtain

$$\begin{aligned} \int W_\sigma(h(ab,b))f(w'n(b,ab,r_1,r_2,r_3,r_4)) \\ \cdot \psi_0(\gamma a^{-1}b^{-2}r_1 + b^{-1}r_2 + r_3)|\gamma^3a^{-3}b^{-5}|d(\ldots) \end{aligned}$$

where

$$n(a,b,c,d,e,f) :=$$
$$x_{0001110}(a)x_{1111110}(b)x_{1112221}(c)x_{1112110}(d)x_{0101110}(e)x_{1112210}(f).$$

From the properties of W_σ we have $|a|, |b| \le 1$. Hence we may conjugate to the right $x_{0001110}(b)x_{1111110}(ab)$ and since f is unramified it is right invariant under these elements. Similarly we may conjugate to the right $w[34561]$ and obtain

$$\int_{|a|,|b|\le 1} W_\sigma(h(ab,b))f(w[76542]n(r_1,r_2,r_3,r_4)) \cdot \psi_0(\gamma a^{-1}b^{-2}r_1 + b^{-1}r_2 + r_3)|\gamma^3 a^{-3} b^{-5}|d(\ldots)$$

where $n(r_1,r_2,r_3,r_4) := x_{0101111}(r_1)x_{0101100}(r_2)x_{0100000}(r_3)x_{0101110}(r_4)$. We conjugate in f the element $x_{-0001100}(t)$ with $|t| \le 1$ from right to left. Changing variables in r_3, we obtain that the integral is zero in the domain $|r_2| > 1$. Hence we get $\int_{|r_2|\le 1}\psi_0(b^{-1}r_2)dr_2$ as inner integration and hence $|b^{-1}| \le 1$. From this and $|b| \le 1$ we get $|b| = 1$. Thus the integral equals

$$\int_{|a|\le 1} W_\sigma(h(a,1))f(w[76542]n(r_1,r_3,r_4))\psi_0(\gamma a^{-1}r_1 + r_3)|\gamma a^{-1}|^3 d(\ldots)$$

where $n(r_1,r_3,r_4) := x_{0101111}(r_1)x_{0100000}(r_3)x_{0101110}(r_4)$. In the same way, by using $x_{-0001110}(t)$ with $|t| \le 1$ and then by $x_{-0001111}(t)$, we may restrict the integration domain to $|r_1| \le 1$ and $|r_4| \le 1$. Hence $|\gamma a^{-1}| \le 1$. Hence the integral equals the product

$$\int_F f(w[76542]x_{0100000}(r_3))\psi_0(r_3)dr_3 \cdot \int_{\substack{|\gamma a^{-1}|\le 1\\ |a|\le 1}} W_\sigma(h(a,1))|\gamma a^{-1}|^3 d^\times a.$$

The first factor is $\mathcal{L}(x\cdot f)$ at $x = 1$, which is 1. What is left to us is the identity

$$W_\pi(diag(\gamma,\gamma,\gamma,1,1,1)) = \int_{|\gamma|\le|a|\le 1} W_\sigma(h(a,1))|\gamma a^{-1}|^3 d^\times a.$$

Integrating both sides over $\gamma \in F^*$ against $|\gamma|^{s-3}$ we obtain (3.13). We are done. ∎

3.4. Main Result. We shall formulate our main result as follows.

Theorem 3.6 (Main): *Let π be an irreducible generic cuspidal automorphic representation of $GSp_3(\mathbb{A})$ with trivial central character. Then the following three statements are equivalent:*

(1) $\pi \subset \Theta_{GE_7}^{GSp_3}(\sigma)$ *for some irreducible generic cuspidal automorphic representation* σ *of* $G_2(\mathbb{A})$.
(2) *The partial spin L-function* $L^S(\pi, Spin(7), s)$ *(see (3.12) or [BG] for a definition) has a simple pole at* $s = 1$.
(3) *The period* $\phi^{GL^{\Delta}(2);\psi_{N_2}}(g)$ *is nonzero for some* $\phi \in \pi$.

Proof: That (2) implies (3) was proved in [V]. To show that (3) implies (1), we first let $\sigma = \Theta_{GE_7}^{G_2}(\pi)$. It follows from Theorems 3.1, 3.2 and 3.3 that σ is a nonzero generic cuspidal automorphic representation of $G_2(\mathbb{A})$. It is clear that $\pi \subset \Theta_{GE_7}^{GSp_3}(\sigma)$, which implies (1) by the cuspidality of both π and σ. Finally, if (1) holds for π, then by Theorem 3.5 we have

$$L^S(\pi, Spin(7), s) = \zeta^S(s) L^S(\sigma, St, s),$$

where $\zeta^S(s)$ is the partial global zeta function and $L^S(\sigma, St, s)$ is the partial standard L-function of G_2 [G2]. Hence $L^S(\pi, Spin(7), s)$ has at least a simple pole at $s = 1$. From [BG] and [V], $L^S(\pi, Spin(7), s)$ can have at most a simple pole at $s = 1$. ∎

3.5. An Analogy for the Lifting from $PGL(3)$ to G_2. An analogy for the lifting from $PGL(3)$ to G_2 will be stated below, the proof of which is similar and will be omitted here. This is an endoscopy lifting. The dual pair correspondence for $PGL(3) \times G_2 \subset PE_6$ was studied in [GRS2]. We have

Theorem 3.7: *Let* σ *be an irreducible generic cuspidal automorphic representation of* $G_2(\mathbb{A})$. *Then the following three statements are equivalent:*
(1) $\sigma \subset \Theta_{PE_6}^{G_2}(\pi)$ *for some cuspidal representation* π *of* $PGL(3, \mathbb{A})$.
(2) *The partial standard L function* $L(\sigma, St, s)$ *(see [G2] for a definition) has a simple pole at* $s = 1$.
(3) *The period* $\varphi^{SL(2);\psi_Z}(h)$ *is nonzero for a certain choice of the data, where*

$$\varphi^{SL(2);\psi_Z}(h) = \int_{SL(2,F)\backslash SL(2,\mathbb{A})} \int_{(F\backslash \mathbb{A})^3} \varphi(z(r_1, r_2, r_3) gh) \psi_Z(r_1) dz dg,$$

and $Z = \{z(r_1, r_2, r_3) = x_{2a+b}(r_1) x_{3a+b}(r_2) x_{3a+2b}(r_3)\}$ *is the abelian unipotent subgroup of* G_2 *generated by the roots* $2a + b$, $3a + b$, *and* $3a + 2b$.

4. D_4-Periods

In this section we shall show the existence of $Spin(8)$-distinguished residual representations of F_4 in terms of the period condition on the cuspidal data. Following [Jng] and [Jng1], those periods are closely related to the existence of the pole of the $Spin(7)$ L-function. Our period on the cuspidal data turns out to be the split version of the one studied in [GS].

4.1. Algebraic Structure of F_4. Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the four simple roots of $G = F_4$ with α_1, α_2 long and α_3, α_4 short. For each simple root α_i, $i = 1, 2, 3, 4$, $\chi_{\alpha_i}(x)$ denotes the one-parameter additive subgroup of G associated to the simple root α_i. Then $SL(2)$ is isomorphic to the subgroup generated by $\chi_{\alpha_i}(x)$ and $\chi_{-\alpha_i}(x)$ and the image of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ is denoted by $h_{\alpha_i}(t)$. The maximal torus T of G is four-dimensional and is generated by $h_{\alpha_i}(t)$ for $i = 1, 2, 3, 4$. Let $B = TU$ be the Borel subgroup determined by this choice of the simple roots. The Weyl group $W = W(G, T)$ is generated by the simple reflections $w_i = w_{\alpha_i}$ for $i = 1, 2, 3, 4$. We will use as in §2 or [B] the notations

$$w[ijk \cdots l] := w_i w_j w_k \cdots w_l, \; ijkl := i\alpha_1 + j\alpha_2 + k\alpha_3 + l\alpha_4.$$

We are going to describe the explicit embedding of the group $Spin(9)$ into F_4. Let β_1, β_2, β_3, and β_4 be the simple roots of $Spin(9)$ with β_4 the short simple root. To describe the embedding, we identify the four simple roots β_i, $i = 1, 2, 3, 4$, with four roots in F_4 as follows:

$$\beta_1 = 0122, \; \beta_2 = 1000, \; \beta_3 = 0100, \; \beta_4 = 0010.$$

Then the positive roots of $Spin(9)$ are written in terms of those of F_4 as follows:

$$\begin{matrix} 1000 & 0100 & 0010 & 0110 & 1100 & 0120 & 1120 & 1220 \\ 1110 & 0122 & 1122 & 1222 & 1232 & 1242 & 1342 & 2342. \end{matrix}$$

By combining the usual way to embed the group $Spin(8)$ into $Spin(9)$, we identify the four simple roots γ_1, γ_2, γ_3, and γ_4 of $Spin(8)$ with those of F_4 as follows:

$$\gamma_1 = 0100, \; \gamma_2 = 1000, \; \gamma_3 = 0120, \; \gamma_4 = 0122.$$

In this way, we see the set of positive roots of $Spin(8)$ in F_4 below:

$$1000,\ 0100,\ 1100,\ 0120,\ 1120,\ 1220,\ 0122,\ 1122,\ 1222,\ 1242,\ 1342,\ 2342.$$

Let $P = MN$ be the standard maximal parabolic subgroup of G associated to the subset of simple roots $\{\alpha_2, \alpha_3, \alpha_4\}$. One knows that M is GSp_3 and N is the Heisenberg group of dimension 15.

LEMMA 4.1: *The generalized flag variety $P\backslash F_4$ decomposes into two $Spin(9)$-orbits with representatives e (the identity) and $\gamma := w[1234]\chi_{0011}(1)$. The stabilizer of e is the maximal parabolic subgroup of $Spin(9)$ with its Levi subgroup $GL(2) \times Spin(5)$ and the stabilizer of γ is $GSp_2 \cdot V$ inside the maximal parabolic subgroup of $Spin(9)$ with its Levi subgroup $GL(4)$, where $GSp_2 \subset GL(4)$ and V is the unipotent radical of the maximal parabolic subgroup.*

PROPOSITION 4.1: *The $Spin(8)$-orbits on the generalized flag variety $P\backslash F_4$ are*

$$\begin{aligned} F_4 =& P\cdot e\cdot Spin(8) \cup P\cdot w[123]\chi_{0010}(1)\cdot Spin(8) \\ &\cup P\cdot w[1234]\chi_{0011}(1)\cdot Spin(8) \cup P\cdot w[1234]\chi_{0011}(1)w[3]\cdot Spin(8) \\ &\cup P\cdot w[1234]\chi_{0011}(1)w[3]\chi_{0010}(1)\cdot Spin(8). \end{aligned}$$

4.2. A RESIDUAL REPRESENTATION OF F_4. Let F be a number field and $\mathbb{A}$ the ring of adeles of F. Let τ be an irreducible cuspidal automorphic representation of $GSp_3(\mathbb{A})$ with trivial central character. We may in addition assume that the representation τ is left A^+-invariant, where the subgroup A^+ is defined in the following decomposition:

$$GSp_3(\mathbb{A}) = U(\mathbb{A})\cdot T(\mathbb{A})^1\cdot A^+\cdot K_M,$$

where $B_M := UT$ is the standard Borel subgroup of $M = GSp_3$ and K_M is the maximal compact subgroup of $GSp_3(\mathbb{A})$.

Let $G := F_4$ be the split group of type F_4 and $K = \prod_v K_v$ be the maximal compact subgroup of $G(\mathbb{A})$ so that $G(\mathbb{A}) = P(\mathbb{A})K$ is the Iwasawa decomposition.

Let A be the (split) center of M. The unique reduced root in $R^+(P,A)$ can be identified with the simple root α_1. As usual, we denote

$$\tilde{\alpha_1} := \langle \rho_P,\ \alpha_1\rangle^{-1}\rho_P, \tag{4.1}$$

where ρ_P is half of the sum of all positive roots in N and $\langle\ ,\ \rangle$ is the usual Killing–Cartan form for the root system. We let

$$\mathfrak{a}_P = Hom_{\mathbb{R}}(X(M),\mathbb{R}),\quad \mathfrak{a}_P^* = X(M)\otimes\mathbb{R}. \tag{4.2}$$

Since P is maximal, $\mathfrak{a}_P^*$ is of dimension one. We identify $\mathbb{C}$ with $\mathfrak{a}_{P,\mathbb{C}}^*$ via $s\mapsto s\tilde{\alpha}_1$.

Let $H_P\colon M\mapsto \mathfrak{a}_P$ be the map defined as follows. For any $\chi\in\mathfrak{a}_P^*$,

$$H_P(m)(\chi) = \prod_v |\chi(m_v)|_v \tag{4.3}$$

for $m \in M(\mathbb{A})$. This map H_P can be extended as a function over $G(\mathbb{A})$ via the Iwasawa decomposition. By direct computation, we know that

$$H_P(m)(s) = |\det m|^{s/3} \quad H_P(m)(\rho_P) = |\det m|^{4/3}. \tag{4.4}$$

Let $\phi(g)$ be a complex-valued smooth function on $G(\mathbb{A})$ which is left $N(\mathbb{A})M(F)$-invariant and right K-finite. For

$$g = pk = nm^1ak \in N(\mathbb{A})M^1A^+K,$$

we assume that

$$\phi(g) = \phi(m^1k). \tag{4.5}$$

If we fix a $k \in K$, the map

$$m^1 \mapsto \phi(m^1k)$$

defines a $K \cap M^1$-finite vector in the space of cusp form τ of $M(\mathbb{A})$. We set

$$F(g;\phi,s) := H_P(g)(s+\rho_P)\phi(g).$$

Attached to such a function $F(g;\phi,s)$, we define an Eisenstein series

$$E(g;\phi,s) := \sum_{\gamma\in P\backslash G} F(\gamma g;\phi,s). \tag{4.6}$$

From the general theory of Eisenstein series [MW], this Eisenstein series converges absolutely for the real part $Re(s) > \frac{4}{3}$ and has a meromorphic continuation to the whole s-plane with finitely many possible simple poles for $Re(s) > 0$.

Suppose that at $s = s_0$ ($0 \le s_0 \le \frac{4}{3}$) the Eisenstein series $E(g;\phi,s)$ has a pole. Let $E_{s_0}(g,\phi)$ be the residue of $E(g;\phi,s)$ at $s = s_0$. We shall compute the period of the residue $E_{s_0}(g,\phi)$ over the spherical subgroup $H = Spin(8)$ by Arthur's truncation method.

4.3. Truncated Formula. We shall use Arthur's truncation formula for Eisenstein series [A] to define the $Spin(8)$-period

$$\int_{H(F)\backslash H(\mathbb{A})} E_{s_0}(h,\phi)dh.$$

The formulation and notations follow from §4 in [Jng]. Hence we have

$$\int_{H(F)\backslash H(\mathbb{A})} E_{s_0}(h,\phi)dh = \mathrm{res}_{s=s_0}[I_1 - I_2] + I_3, \tag{4.7}$$

where, for $j = 1, 2, 3$,

$$I_j := \int_{H(F)\backslash H(\mathbb{A})} \theta_j(h)dh, \tag{4.8}$$

and

$$\theta_1(g) := \sum_{\gamma\in P\backslash G} F(\gamma g, \phi, s)(1 - \tau_c(H(\gamma g)));$$

$$\theta_2(g) := \sum_{\gamma\in P\backslash G} F(\gamma g; M(s)(\phi), -s)\tau_c(H(\gamma g));$$

$$\theta_3(g) := \sum_{\gamma\in P\backslash G} F(\gamma g; M_{s_0}(\phi), -s_0)\tau_c(H(\gamma g)).$$

From Proposition 4.1, the stabilizer $P\cap H$ of the closed orbit PeH is a parabolic subgroup in H. We set

$$P\cap H := P_2 = M_2N_2.$$

The parabolic subgroup P_2 of H is the one whose Levi subgroup M_2 is generated by three simple roots $\gamma_1, \gamma_3, \gamma_4$ of H.

PROPOSITION 4.2: *The integrals I_j, for $j = 1, 2, 3$, defined in (4.8), absolutely converge and satisfy the following relations,*

$$I_1 = \frac{c^{\frac{s-1}{3}}}{s-1}\int_{[M_2(F)\backslash M_2^1]\times K_H} \phi(m^1k)dm^1dk$$

and $I_3 = \mathrm{res}_{s=s_0} I_2$.

This Proposition will be proved in subsection 4.4. By formula (4.7), we have

THEOREM 4.1: *Let s_0 be a real number with $0 \le s_0 \le \frac{4}{3}$. If $s_0 \ne 1$, the residual representation $E_{s_0}(g, \phi)$ of the Eisenstein series $E(g; \phi, s)$ at $s = s_0$ is not H-distinguished if it exists. If $s_0 = 1$, the residual representation $E_1(g, \phi)$ of the Eisenstein series $E(g; \phi, s)$ at $s = 1$ is nonzero and H-distinguished if and only if the integral*

$$\int_{[M_2(F)\backslash M_2^1]\times K_H} \phi(m^1k)dm^1dk$$

does not vanish. Moreover, we have

$$\int_{H(F)\backslash H(\mathbb{A})} E_1(h, \phi)dh = \int_{[M_2(F)\backslash M_2^1]\times K_H} \phi(m^1k)dm^1dk.$$

Proof: The case of $s_0 \ne 1$ follows directly from Proposition 4.1 and formula (4.7). We shall only consider the case of $s_0 = 1$.

In this case, by formula (4.7) again, the residual representation $E_1(g,\phi)$ is H-distinguished if and only if the residue at $s=1$ of the integral I_1 does not vanish, which is equivalent to that the integral

$$\int_{[M_2(F)\backslash M_2^1]\times K_H} \phi(m^1k)dm^1dk$$

does not vanish. Finally, the identity is clear. ∎

THEOREM 4.2: *The residue at $s=1$ of $E(g;\phi,s)$ on F_4 is nonzero and $H = Spin(8)$-distinguished if and only if the irreducible cuspidal representation τ (the cuspidal data) of $M = GSp_3$ is $H' = [GL(2)^3]^\circ$-distinguished.*

Proof: First of all, we notice that the image of the embedding of M_2 in $M = GSp_3$ is

$$[GL(2)^3]^\circ = \{(g_1,g_2,g_3)\in GL(2)^3\colon \det g_1 = \det g_2 = \det g_3\}.$$

The inner integral

$$\begin{aligned}\int_{M_2(F)\backslash M_2^1}\phi(m^1)dm^1 &= \int_{[GL(2)^3](F)\backslash[GL(2)^3]^1}\phi(g^1)dg^1\\ &= \frac{\mathrm{vol}(\mathbb{A}^1/F^\times)}{2}\cdot\int_{Z_M(\mathbb{A})[GL(2)^3](F)\backslash[GL(2)^3]^\circ(\mathbb{A})}\phi(g)dg.\end{aligned}$$

It is easy to see that if the residue at $s=1$ of $E(g;\phi,s)$ on F_4 is $H = Spin(8)$-distinguished, i.e.

$$\int_{H(F)\backslash H(\mathbb{A})} E_1(h,\phi)dh \neq 0$$

for some smooth function ϕ on $G(\mathbb{A})$ as defined in (4.7), then for the given smooth function ϕ, the inner period

$$\int_{Z_M(\mathbb{A})[GL(2)^3](F)\backslash[GL(2)^3]^\circ(\mathbb{A})}\phi(g)dg \neq 0.$$

Since $m\mapsto\phi(m)$ is a cusp form in τ, this integral is well defined. Hence the irreducible cuspidal representation τ is H'-distinguished.

To prove the other direction of the implication, we need the argument used in the proof for Theorem 3.2 in [GJR].

For an irreducible cuspidal automorphic representation τ, one can have a smooth function ϕ on $G(\mathbb{A})$ as defined in (4.5) and also one can define a smooth function Φ on K (the maximal compact subgroup in $G(\mathbb{A})$) with values in the space of τ, satisfying condition

$$\Phi(pk) = \tau(m)\Phi(k)$$

when $p = nm \in P(\mathbb{A}) \cap K$.

If τ is H'-distinguished, then there is a smooth function ϕ as in (4.5) such that the integral

$$\mathcal{P}(\phi) := \int_{M_2(F)\backslash M_2^1} \phi(m^1)dm^1$$

does not vanish. Note that this integral defines a continuous functional over the space of smooth functions as defined in (4.5). It is easy to see by restriction to K that there is a nonzero smooth function Φ as defined above such that

$$I(\Phi) = \int_K \mathcal{P}(\Phi(k))dk = \int_K \int_{M_2(F)\backslash M_2^1} \phi(m^1 k)dm^1 dk.$$

The point here is to show that $I(\Phi)$ is a nonzero functional, which means that the nonzero functional over $M(\mathbb{A})$ extends to a nonzero functional over $G(\mathbb{A})$. Assume that there is a factorizable function $\phi_\tau \in \tau$ such that

$$\mathcal{P}(\phi_\tau) \neq 0.$$

Write $\phi_\tau = \phi_S \otimes \phi^S$, where ϕ^S is the infinite tensor product of all unramified local components of ϕ_τ and ϕ_S is the finite tensor product of all archimedean or ramified local components of ϕ_τ. Since the functional $\mathcal{P}$ is continuous, there are continuous functions $\mathcal{P}_S$ and $\mathcal{P}^S$ over the spaces of smooth vectors in τ_S and τ^S, respectively, such that

$$\mathcal{P}(\phi_\tau) = \mathcal{P}_S(\phi_S) \cdot \mathcal{P}^S(\phi^S).$$

Now since ϕ^S is unramified, one can naturally take

$$\Phi^S(k^S) = \phi^S$$

for all $k^S \in K^S := \prod_{v \notin S} K_v$. Then we have

$$\mathcal{P}^S(\Phi^S(k^S)) = \mathcal{P}^S(\phi^S).$$

Since S is finite, one may assume that

$$\mathcal{P}_S(\Phi_S) = \prod_{v \in S} \mathcal{P}_v(\Phi_v).$$

By the admissibility of the local components of τ, it follows from the standard argument used in [JR] and [Jng] that there is a smooth function Φ_S such that

$$I_S(\Phi_S) = \int_{K_S} \mathcal{P}_S(\Phi_S(k_S))dk_S \neq 0$$

where $K_S := \prod_{v\in S} K_v$. Finally, we take $\Phi = \Phi_S \otimes \Phi^S$ such that

$$I(\Phi) = I_S(\Phi_S) \cdot I^S(\Phi^S) \neq 0.$$

We are done. ∎

From the proof, we obtain the following consequence, which is important for further applications of such periods [GP] and [GK].

COROLLARY 4.1: *Let τ be an irreducible cuspidal automorphic representation of $GSp_3(\mathbb{A})$ with trivial central character. If τ is H'-distinguished, then for almost all unramified local components τ_v, there is an H'_v-invariant functional which takes nonzero value at unramified vectors in τ_v.*

Remark 4.1: In general, for a given separable cubic commutative algebra E over F, there is an algebraic group D_4^E of D_4-type in the split exceptional group F_4. Our set-up for the comparison of the 'outer' D_4^E-period on F_4 with the 'inner' $R_{E/F}(A_1)$-period on $GSp_3(\mathbb{A})$ still makes sense. The relation of such periods to the spin L-functions can be expected following the argument in [Jng1]. The local version of such inner periods has been studied by B. Gross and G. Savin to find a motive with Galois group G_2 [GS].

4.4. EXPLICIT COMPUTATION OF INTEGRALS. We shall compute integrals of the type

$$I = \int_{H(F)\backslash H(\mathbb{A})} \theta(h)dh,$$

with the assumption of its convergence, where

$$\theta(g) = \sum_{\gamma\in P\backslash G} F(\gamma g)$$

for suitable left $P(F)$-invariant functions F on $G(\mathbb{A})$. In particular, we consider $\theta(g) = \theta_i(g)$, $i = 1, 2, 3$, respectively. The computation yields a proof of Proposition 4.2. The verification of the convergence of integrals involved in the computation should be carried out as in §7 in [Jng] and will be omitted here. From now on we assume all integrals involved here are convergent.

First of all, by Lemma 4.1, we have

$$\begin{aligned} I &= \sum_{\gamma\in P\backslash P\cdot Spin(9)/H} \int_{H^\gamma(F)\backslash H(\mathbb{A})} F(\gamma h)dh \\ &= \sum_{\gamma\in P\backslash P\mu Spin(9)/H} \int_{H^\gamma(F)\backslash H(\mathbb{A})} F(\gamma h)dh \\ &= J_1 + J_2, \end{aligned} \tag{4.9}$$

where $\mu = w(1234)\chi_{0011}(1)$.

Since $P \cap Spin(9) = P_{GL(2)\times Spin(5)}$, the parabolic subgroup of $Spin(9)$ with the Levi subgroup $GL(2) \times Spin(5)$, integral J_1 can be expressed as

$$J_1 = \sum_{\gamma\in P_{GL(2)\times Spin(5)}\backslash Spin(9)/H} \int_{H^\gamma(F)\backslash H(\mathbb{A})} F(\gamma h)dh.$$

By Proposition 2.1, we have

$$P_{GL(2)\times Spin(5)}\backslash Spin(9)/H = [P_{GL(2)\times Spin(5)}eH] \cup [P_{GL(2)\times Spin(5)}\delta H],$$

where $\delta = w(123)\chi_{0010}(1)$, and

$$\delta^{-1}P_{GL(2)\times Spin(5)}\delta \cap H = (GL_1 \times Spin(5)) \cdot V_{1,3},$$

where $(GL_1 \times Spin(6)) \cdot V_{1,3}$ is a maximal parabolic subgroup of $H = Spin(8)$ and $Spin(5)$ is naturally embedded into $Spin(6)$. From this, we deduce that

$$J_1 = \int_{[P_{GL(2)\times Spin(5)}\cap H](F)\backslash H(\mathbb{A})} F(h)dh = \int_{[(GL(1)\times Spin(5))\cdot V_{1,3}](F)\backslash H(\mathbb{A})} F(\delta h)dh.$$

We claim that the second integral is zero. In fact, since $\delta \cdot Spin(5) \cdot \delta^{-1} = Sp_2$, which is a subgroup of Sp_3, and also $\delta \cdot \chi_{1242}(x) \cdot \delta^{-1} = \chi_{0122}(x)$, the integral

$$\int_{[(GL(1)\times Spin(5))\cdot V_{1,3}](F)\backslash H(\mathbb{A})} F(\delta h)dh$$

has an inner integral of type

$$\int_{Sp_2(F)\backslash Sp_2(\mathbb{A})} \int_{F\backslash\mathbb{A}} F(yx\delta h)dxdy.$$

By the definition of $F(g)$, for a fixed h, the function $F(m\delta h)$ is a cusp form in m over the Levi subgroup GSp_3. Then the inner integral can be viewed as

$$\int_{Sp_2(F)\backslash Sp_2(\mathbb{A})} \int_{F\backslash\mathbb{A}} \phi(\begin{pmatrix} 1 & 0 & x \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix})dxdy \tag{4.10}$$

for a cusp form ϕ over GSp_3.

LEMMA 4.2: *For any irreducible cuspidal automorphic representation τ of $GSp_3(\mathbb{A})$ and any cusp form $\phi_\tau \in \tau$, the period defined by (4.10) always vanishes.*

Proof: We expand along the Heisenberg unipotent radical modulo its center. The Sp(2) acts on the group by two orbits. It can be checked that in each orbit we obtain as a stabilizer a unipotent radical of Sp(3). By cuspidality of π the lemma follows. ∎

By Lemma 4.2, we have the following:

$$J_1 = \int_{[P_{GL(2)\times Spin(5)} \cap H](F)\backslash H(\mathbb{A})} F(h)dh. \tag{4.11}$$

Next we are going to show that integral J_2 is zero. Recall that

$$J_2 = \sum_{\gamma \in P\backslash P\mu Spin(9)/H} \int_{H^\gamma(F)\backslash H(\mathbb{A})} F(\gamma h)dh$$

where $\mu = w(1234)\chi_{0011}(1)$. It is not difficult to check that

$$\mu^{-1}P\mu \cap Spin(9) = GSp(4)\cdot V_{9,4}$$

where $GSp_2 \subset GL(4)$ and $V_{9,4}$ is the unipotent radical of the standard maximal parabolic subgroup of $Spin(9)$ with $GL(4)$ Levi subgroup. To compute integral J_2, one needs the decomposition of $GSp_2 \cdot V_{9,4}\backslash Spin(9)/H$. By direct computation, one knows that the space $GSp(4)\cdot V_{9,4}\backslash Spin(9)/H$ decomposes into three different double cosets with representatives

$$e, \quad w(3), \quad w(3)\chi_{0010}(1),$$

respectively. For representative e, the integral is

$$I_e := \int_{[GSp_2\cdot V_{8,4}](F)\backslash H(\mathbb{A})} F(\mu h)dh$$

where $V_{8,4} = V_{9,4}\cap H$. Since $\mu^{-1}\chi_{1242}(x)\mu = \chi_{0122}(x)$, the above integral has an inner integral of the following type:

$$\int_{Sp_2(F)\backslash Sp_2(\mathbb{A})}\int_{F\backslash\mathbb{A}} \phi(\begin{pmatrix} 1 & 0 & x \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix})\psi(x)dxdy. \tag{4.12}$$

LEMMA 4.3: *For any irreducible cuspidal automorphic representation τ of $GSp_3(\mathbb{A})$ and any cusp form $\phi_\tau \in \tau$, the period defined by (4.12) always vanishes.*

Proof: Using [I], the integral in (4.12) is zero for all choices of data if and only if the integral

$$\int_{[Sp_2N_1](F)\backslash[Sp_2N_1](\mathbb{A})} f(ug)\tilde{\theta}(ug)\tilde{\theta}(g)dudg$$

is zero for all choices of data, where Sp_2N_1 is a Jacobi group with

$$N_1 = \{\begin{pmatrix} 1 & u & x \\ 0 & I_4 & u^* \\ 0 & 0 & 1 \end{pmatrix}\}$$

and $y \in Sp_2$ is embedded as y in (4.12), and $\tilde{\theta}(g)$ is the usual theta function on the double cover of Sp_2N_1. Replacing $\tilde{\theta}(g)$ by the Siegel Eisenstein series and unwinding the integral for Re(s) large, we know that the integral is zero for all choices of data. Since $\tilde{\theta}(g)$ is the residue of the Siegel Eisenstein series, this implies that the above integral is zero for all choices of data. ∎

By Lemma 4.3, integral I_e vanishes. For representative $w(3)$, one has the similar stabilizer since the Weyl group element $w(3)$ takes the parabolic to its associated one. Therefore the integral attached to $w(3)$,

$$\int_{[GSp_2\cdot V_{8,4}]^{w(3)}(F)\backslash H(\mathbb{A})} F(\mu w(3)h)dh,$$

also vanishes.

Finally, we are going to consider the open orbit case which is represented by $w(3)\chi_{0010}(1)$. The integral is

$$\int_{[GSp_2\cdot V_{8,4}]^{w(3)\chi_{0010}(1)}(F)\backslash H(\mathbb{A})} F(\mu w(3)\chi_{0010}(1)h)dh.$$

One can see that $[GSp_2 \cdot V_{8,4}]^{w(3)\chi_{0010}(1)} = GL(2) \cdot V^9$, where $GL(2)$ is embedded into $H = Spin(8)$ as

$$g \mapsto \begin{pmatrix} |g| & & & & & \\ & g & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & g^* & \\ & & & & & |g|^{-1} \end{pmatrix},$$

which is generated by the simple root 1000 in F_4, where $|g| = \det g$, and V^9 is a 9-dimensional unipotent subgroup of H of following type:

$$\begin{pmatrix} 1 & x & y & z & r_1 & r_2 & r_3 & 0 \\ & 1 & 0 & y & r_4 & r_5 & 0 & \\ & & 1 & -x & r_6 & 0 & * & \\ & & & 1 & 0 & & & \\ & & & & 1 & & & \\ & & & & & 1 & * & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}.$$

On the other hand, we have

$$\begin{aligned} \mu w(3)\chi_{0010}(1) =& w(12343)\chi_{0010}(1)\chi_{0011}(1) \\ =& w(41234)\chi_{0010}(1)\chi_{0011}(1) \\ =& w(4)\chi_{0001}(1)w(1234)\chi_{0011}(1). \end{aligned}$$

Since $w(4)\chi_{0001}(1) \in P(F)$, the integral reduces to

$$\int_{[GL(2)\cdot V^9](F)\backslash H(\mathbb{A})} F(w(1234)\chi_{0011}(1)h)dh.$$

By factorizing through the subgroup $GL(2) \cdot V^9$, we get an inner integral of the following type:

$$(4.13) \int_{GL(2)(F)\backslash GL(2)(\mathbb{A})} \cdot \int_{F\backslash\mathbb{A}} \phi(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & r \\ & 1 & x & y & z & 0 \\ & & 1 & 0 & y & 0 \\ & & & 1 & -x & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} |g| & & & & \\ & |g| & & & \\ & & g & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix})dgd(\cdots).$$

Assumption 4.1: *We claim that the integral defined in (4.13) must be zero as long as the residue $E_1(h, \phi_\tau)$ is not zero. In fact, after a certain Fourier expansion, the integral in (4.13) equals the period integral defined in Theorem 2.1 in* [GRS3]. *Then Theorem 2.1 in* [GRS3] *implies that if the integral defined in (4.13) does not vanish for a certain choice of data, then the partial standard L-function $L^S(s, \tau, St)$ has a simple pole at $s = 1$. By the recent work* [BFG], *this implies that the partial spin L-function $L(s, \tau, Spin(7))$ is holomorphic at $s = 1$. Assume now that the complete spin L-function defined by the Rankin–Selberg method in*

[BG] *is the same as the one defined by the Langlands–Shahidi method in* [Sh]. *Then, by the conjecture of the normalization of local intertwining operators by a product of relevant L-functions, one should obtain that $E_1(h, \phi_\tau)$ is zero. In this paper, we assume that the integral in (4.13) vanishes for all given data.*

By taking $\theta(g) = \theta_j(g)$, $j = 1, 2, 3$, respectively, we obtain, by assuming the convergence of all integrals, that

$$I_1 = \int_{[P_{GL(2)\times Spin(5)}\cap H](F)\backslash H(\mathbb{A})} F(h, \phi, s)(1 - \tau_c(H(h)))dh,$$
$$I_2 = \int_{[P_{GL(2)\times Spin(5)}\cap H](F)\backslash H(\mathbb{A})} F(h, M(s)(\phi), -s)\tau_c(H(h))dh,$$
$$I_3 = \int_{[P_{GL(2)\times Spin(5)}\cap H](F)\backslash H(\mathbb{A})} F(h, M_{s_0}(\phi) - s_0)\tau_c(H(h))dh.$$

Following the decomposition of $G(\mathbb{A})$,

$$G(\mathbb{A}) = N(\mathbb{A})M^1 A^+ K,$$

we have a similar decomposition for $H(\mathbb{A})$ with respect to the parabolic subgroup $P_{GL(2)\times Spin(5)} \cap H = P_2$. Then we have

$$I_1 = \int_{K_H\times[M_2(F)\backslash M_2^1]} \phi(m^1k)dm^1dk \int_{A^+} H(a)^{(s+4-5)/3}(1 - \tau_c(H(a)))da^\times.$$

Since we have

$$\int_{A^+} H(a)^{(s+4-5)/3}(1 - \tau_c(H(a)))da^\times = \int_0^{c^{1/3}} t^{s-2}dt = \frac{c^{(s-1)/3}}{s-1},$$

we obtain

$$I_1 = \frac{c^{(s-1)/3}}{s-1} \cdot \int_{K_H\times[M_2(F)\backslash M_2^1]} \phi(m^1k)dm^1dk. \tag{4.14}$$

By the same calculation, we obtain

$$I_2 = \frac{c^{\frac{-s-1}{3}}}{s+1} \cdot \int_{K_H\times[M_2(F)\backslash M_2^1]} M(s)(\phi)(m^1k)dm^1dk \quad (\mathrm{Re}(s) > -1),$$
$$I_3 = \frac{c^{\frac{-s_0-1}{3}}}{s_0+1} \cdot \int_{K_H\times[M_2(F)\backslash M_2^1]} M_{s_0}(\phi)(m^1k)dm^1dk.$$

This proves Proposition 4.2. ∎

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